Petri Nets (for Planners)

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... from various places ...

ICAPS 2011
Petri Nets (PNs) is a formalism for modelling discrete event systems.

- Developed by (and named after) C.A. Petri in 1960s

- In general Petri nets, places are **unbounded counters**
  - advantages in expressivity and modelling convenience
  - questions of reachability, coverability, etc. are computationally harder to answer, but still decidable
Exchange of ideas between Petri nets and planning holds potential to benefit both areas:

- Analysis methods for Petri nets are often based on ideas & techniques not common in planning:
  - algebraic methods based on the state equation
  - rich literature on the study of classes of nets with special structure

- Yet, some standard planning techniques (e.g., search heuristics) are unknown in the PN community
Outline of the Tutorial

1. Definitions, notation and modelling
2. Decision problems, complexity and expressivity
3. Analysis techniques for general Petri nets
   - Coverability
   - The state equation
   - Reachability
4. Petri nets with special structure
5. Conclusions
Definitions, Notation and Modelling
A Petri net has **places**, **transitions**, and **directed arcs**

Arcs connect places to transitions or vice versa

Places contain zero or finite number of **tokens**

A **marking** is disposition of tokens in places

A transition is **fireable** if there is token at the start place of each input arc

When transition fires:
- it **consumes** token from start place of each input arc
- it **puts** token at end place of each output arc

Execution is **non-deterministic**
Example
Formal Definition

Place/Transition (P/T) net is tuple \( N = (P, T, W) \) where:

- \( P \) is set of places
- \( T \) is set of transitions (and \( P \cap T = \emptyset \))
- \( W \subseteq (P \times T) \cup (T \times P) \rightarrow \mathbb{N} \)
  (multiset of arcs: each \((x, y)\) has multiplicity \( W(x, y) \))

For transition \( t \):

- **preset** is \( \cdot t = \{ s : W(s, t) > 0 \} \) (input places)
- **postset** is \( t^\cdot = \{ s : W(t, s) > 0 \} \) (output places)

**Marking** is \( m : P \rightarrow \mathbb{N} \) (zero or more tokens at each place)
Example

- marking $m = \langle 1, 1, 0, 0, 0, 0, 0 \rangle$
- transition $t_6$: $\bullet t_6 = \{p_3, p_5\}$, $t_6^* = \{p_6, p_7\}$
A transition $t$ is **enabled** or **firable** at marking $m$ if

$$m(p) \geq W(p, t) \quad \text{for each } p \in \bullet t$$
A transition $t$ is **enabled** or **firable** at marking $m$ if

$$m(p) \geq W(p, t) \quad \text{for each } p \in \dot{t}$$

Upon firing, $t$ **produces** new marking $m'$ such that

$$m'(p) = \begin{cases} 
  m(p) - \underbrace{W(p, t)}_{\text{consumed}} + \underbrace{W(t, p)}_{\text{added}} & \text{if } p \in \dot{t} \cup t^* \\
  m(p) & \text{if } p \notin \dot{t} \cup t^*
\end{cases}$$
Transition relations:

- $m [t] m'$ if $t$ is enabled at $m$ and produces $m'$
- $m [\sigma] m'$, for sequence $\sigma = t_1 t_2 \cdots t_n$, if exists $m''$ with
  - $m [t_1] m''$
  - $m'' [\sigma'] m'$ for $\sigma' = t_2 \cdots t_n$
- $m \not\! [t] m'$ if $m [t] m'$ for some $t$
- $[\ast]$ is transitive closure of $[\ ]$
Execution Semantics

**Transition relations:**

- $m \langle t \rangle m'$ if $t$ is enabled at $m$ and produces $m'$
- $m \langle \sigma \rangle m'$, for sequence $\sigma = t_1 t_2 \cdots t_n$, if exists $m''$ with
  - $m \langle t_1 \rangle m''$
  - $m'' \langle \sigma' \rangle m'$ for $\sigma' = t_2 \cdots t_n$
- $m \langle \rangle m'$ if $m \langle t \rangle m'$ for some $t$
- $\langle * \rangle$ is transitive closure of $\langle \rangle$

**Marked net** is $(N = (P, T, W), m_0)$ where $m_0$ is initial marking
**Execution Semantics**

**Transition relations:**
- \( m [t] m' \) if \( t \) is enabled at \( m \) and produces \( m' \)
- \( m [\sigma] m' \), for sequence \( \sigma = t_1 t_2 \cdots t_n \), if exists \( m'' \) with
  - \( m [t_1] m'' \)
  - \( m'' [\sigma'] m' \) for \( \sigma' = t_2 \cdots t_n \)
- \( m \sigma m' \) if \( m [t] m' \) for some \( t \)
- \( [\ast] \) is transitive closure of \( [\ast] \)

**Marked net** is \( (N = (P, T, W), m_0) \) where \( m_0 \) is initial marking

**Reachable markings** \( R(N, m_0) = \{ m \in \mathbb{N}^P : m_0 [\ast] m \} \)
Execution Semantics

**Transition relations:**

- $m \mathrel{[t]} m'$ if $t$ is enabled at $m$ and produces $m'$
- $m \mathrel{[σ]} m'$, for sequence $σ = t_1 t_2 \cdots t_n$, if exists $m''$ with
  - $m \mathrel{[t_1]} m''$
  - $m'' \mathrel{[σ']} m'$ for $σ' = t_2 \cdots t_n$
- $m \mathrel{[\top]} m'$ if $m \mathrel{[t]} m'$ for some $t$
- $[\ast]$ is transitive closure of $[\top]$

**Marked net** is $(N = (P, T, W), m_0)$ where $m_0$ is initial marking

**Reachable markings** $R(N, m_0) = \{ m \in \mathbb{N}^P : m_0 [\ast] m \}$

**Firing sequences** $L(N, m_0) = \{ σ \in T^{<\infty} : \exists m. m_0 [σ] m \}$
Example

- **marking** \( m = \langle 1, 1, 0, 0, 0, 0, 0 \rangle \)
- **enabled**: \( t_1, t_2, t_3 \)
Example

- **marking** \( m = \langle 0, 1, 1, 0, 0, 0, 0, 0 \rangle \)
- **enabled**: \( t_3 \)
- **marking** \( m = \langle 0, 0, 1, 0, 2, 0, 0 \rangle \)
- **enabled**: \( t_6 \)
Example

- **marking** $m = \langle 0, 0, 0, 0, 1, 1, 1 \rangle$
- **enabled**: $t_4, t_5$
Arithmetic of Functions

For two functions \( f, g \in \mathbb{N}^X \):

- \( f \geq g \) if \( f(x) \geq g(x) \) for each place \( x \)
- \( f > g \) if \( f \geq g \) and there is \( x \) such that \( f(x) > g(x) \)
- \( f + g \) defined pointwise as \( (f + g)(x) = f(x) + g(x) \)

Hence, \( \mathbf{m}[t] \mathbf{m}' \) iff

\[
\begin{align*}
\mathbf{m} & \geq W(\cdot, t) \quad \text{(enable condition)} \\
\mathbf{m}' &= \mathbf{m} - W(\cdot, t) + W(t, \cdot)
\end{align*}
\]
A P/T net $N = (P, T, W)$ is **ordinary** iff $W(p, t) \leq 1$ for all $p, t$.

**Thm:** any net can be transformed into **equivalent** ordinary net.
Transformation Rules

1) $p^{2k} \rightarrow t \Rightarrow p^k \rightarrow t^k$

2) $p^{2k+1} \rightarrow t \Rightarrow p^k \rightarrow t^k + 1$

3) $t^{2k} \rightarrow p \Rightarrow t^k \rightarrow p^k$

4) $t^{2k+1} \rightarrow p \Rightarrow t^k \rightarrow p^k + 1$

Each rule decrease multiplicity by half and add 2 nodes.

Resulting size is $O(\sum_{x,y} W(x, y))$ (exponential)
Example
Example
Example
Example

\[ p \rightarrow t \rightarrow \circ \rightarrow \square \rightarrow \circ \]

\[ q \rightarrow \square \rightarrow \circ \rightarrow \square \rightarrow q \]
Types of Nets

- Marking $m$ is $k$-bounded if $m(p) \leq k$ for all $p \in P$

- Marked net $(N, m_0)$ is $k$-bounded if every reachable marking is $k$-bounded

- It is bounded if it is $k$-bounded for some $k$

- It is safe if it is 1-bounded
Safe Networks

- Every reachable marking is 1-bounded

- Marking $m$ can be thought as state where places represents fluents:
  - if $m(p) = 1$ then fluent $p$ is true at $m$
  - if $m(p) = 0$ then fluent $p$ is false at $m$

- Safe networks can be used for STRIPS planning
Direct STRIPS to PN Translations

- Each atom is a place

- Each (grounded) action is a transition $t$:
  - input arcs $p \rightarrow t$ for each precondition $p$
  - output arcs $t \rightarrow p$ for each positive effect $p$
  - output arcs $t \rightarrow p$ for each precondition $p$ that is not deleted nor added

- Initial state gives initial marking

- Goal state gives \textbf{partial} desired marking

- Plan existence becomes ”Coverability” problem
Example: Gripper w/ 1 Ball and 2 Arms

-move2B, pick1@B, move2A, drop1@A
Example: Gripper w/ 1 Ball and 2 Arms

move2B, pick1@B, move2A, drop1@A
Example: Gripper w/ 1 Ball and 2 Arms

move2B, pick1@B, move2A, drop1@A
Example: Gripper w/ 1 Ball and 2 Arms

move2B, pick1@B, move2A, drop1@A
Example: Gripper w/ 1 Ball and 2 Arms

move2B, pick1@B, move2A, drop1@A
Safe STRIPS Problems

STRIPS problem is safe if its direct translation \((N, m_0)\) is safe

Sufficient Condition:

- For each added atom \(p\), there is precondition \(q\) that is deleted such that \(\{p, q\}\) is mutex

Enforcing the condition:

- Add ‘not-\(p\)’ atoms for each atom \(p\)

- For each action that contains a deleted atom \(p\) that is not precondition, generate two similar actions with \(p\) and not-\(p\) in precondition (respectively)

- Worst-case size of pre-processing is exponential in number of atoms that are deleted and don’t appear as preconditions
Modelling Planning Problems

General nets can “store” multiple tokens at single place

Places can be used to represent:

- **number** of identical objects at location
- **resource quantity**
Gripper with 3 identical balls and 2 identical arms

pick@A, pick@A, move2B, drop@B, drop@B, move2A, pick@A, move2B, drop@B
Gripper with 3 identical balls and 2 identical arms

<table>
<thead>
<tr>
<th>Robby@A</th>
<th># in Room A</th>
<th># held</th>
</tr>
</thead>
<tbody>
<tr>
<td>pick@A</td>
<td># held</td>
<td></td>
</tr>
<tr>
<td>drop@A</td>
<td># free</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Robby@B</th>
<th># in Room B</th>
</tr>
</thead>
<tbody>
<tr>
<td>move2A</td>
<td># held</td>
</tr>
<tr>
<td>pick@B</td>
<td># held</td>
</tr>
<tr>
<td>drop@B</td>
<td># free</td>
</tr>
</tbody>
</table>

pick@A, **pick@A**, move2B, drop@B, drop@B, move2A, pick@A, move2B, drop@B
Gripper with 3 identical balls and 2 identical arms

pick@A, pick@A, **move2B**, drop@B, drop@B, move2A, pick@A, move2B, drop@B
Gripper with 3 identical balls and 2 identical arms

pick@A, pick@A, move2B, drop@B, drop@B, move2A, pick@A, move2B, drop@B
Gripper with 3 identical balls and 2 identical arms

pick@A, pick@A, move2B, drop@B, drop@B, move2A, pick@A, move2B, drop@B
Gripper with 3 identical balls and 2 identical arms

pick@A, pick@A, move2B, drop@B, drop@B, move2A, pick@A, move2B, drop@B
Gripper with 3 identical balls and 2 identical arms

Robby@A

# in Room A

# held

pick@A

# free

drop@A

Robby@B

# in Room B

move2B

move2A

pick@A

don't have

pick@B

drop@B

don't have

drop@B

pick@A, pick@A, move2B, drop@B, drop@B, move2A, pick@A, move2B, drop@B
Gripper with 3 identical balls and 2 identical arms

pick@A, pick@A, move2B, drop@B, drop@B, move2A, pick@A, move2B, drop@B
Gripper with 3 identical balls and 2 identical arms

pick@A, pick@A, move2B, drop@B, drop@B, move2A, pick@A, move2B, drop@B
Gripper with 3 identical balls and 2 identical arms

pick@A, pick@A, move2B, drop@B, drop@B, move2A, pick@A, move2B, drop@B
Wargus Domain (Chan et al. 2007)
Other Types of Nets

**State Machines:**
- every **transition** has one incoming and one outgoing arc
  i.e. $|\cdot t| = |t\cdot| = 1$ for each $t \in T$

**Marked Graphs:**
- every **place** has one incoming arc, and one outgoing arc
  i.e. $|\cdot p| = |p\cdot| = 1$ for each $p \in P$

**Free-choice Nets:**
- every arc is either the **only arc going from** the place, or
  **only arc going to** the transition
  i.e. $|p\cdot| \leq 1$ or $\cdot(p\cdot) = \{p\}$ for each $p \in P$
Extensions

**Inhibitor arcs** (enablers):
- transition enabled when there is **no token at place**

**Read arcs** (enablers):
- do not **consume** tokens

**Reset arcs**: erase all tokens at place

**Others**:
- colored, hierarchical, prioritization, ...
Two vectors associated with transition $t$:

$$
W_t^- = \left( \begin{array}{c}
W(p_1, t) \\
\vdots \\
W(p_{|P|}, t)
\end{array} \right) \quad W_t^+ = \left( \begin{array}{c}
W(t, p_1) \\
\vdots \\
W(t, p_{|P|})
\end{array} \right)
$$

- $t$ enabled at $m$ iff $m \geq W_t^-$
- $W_t = W_t^+ - W_t^-$ is effect of $t$
- firing $t$ leads to $m' = m + W_t$
- $W = \left( W_{t_1}, W_{t_2}, \ldots, W_{t_{|T|}} \right)$ is incidence matrix
- $r_p$: row of $W$ corresponding to place $p$
Examples

\[ W = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \]

\[ W = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]
Pure nets have no “self loops”: $\bullet t \cap t^* = \emptyset$ for every transition $t$.

For pure nets, incidence matrix $W$ unambiguously defines the net.

Any net can be transformed into a pure net by splitting loops:

Transformation is linear space.
Complexity & Expressivity
Decision Problems for Marked Nets

Given a marked net $(N, m_0)$:

- **Reachability**: Is there a firing sequence that ends with given marking $m$?

- **Coverability**: Is there a firing sequence that ends with marking $m'$ such that $m' \geq m$ for given $m$?

- **Boundedness**: Does there exist an integer $k$ such that every reachable marking is $k$-bounded? $m \leq K$?

Coverability and boundedness are EXPSPACE-complete

Reachability is EXPSPACE-hard, but existing algorithms are non-primitive recursive (i.e., have unbounded complexity)
More Properties

- **Executability**: Is there a firing sequence valid at $m_0$ that includes transition $t$?
  - Reduces to coverability: $t$ is executable iff $W_t$ is coverable
  - and vice versa: reduction using a “goal transition”

- **Repeated Executability**: Is there a firing sequence in which a given transition (or set of transitions) occurs an infinite number of times?

- **Reachable Deadlock**: Is there a reachable marking $m$ at which no transition is enabled?

- **Liveness**: Executability of every transition at every reachable marking, i.e.,
  $$\forall M : M_0 \xrightarrow{\star} M \rightarrow \forall t \exists M', M'' : M \xrightarrow{\star} M' [t] M''.$$  

...and many more ...
Equivalence Problems

- **Equivalence**: Given two marked nets, \((N_1, m_1)\) and \((N_2, m_2)\), with equal (or isomorphic) sets of places, do they have the equal sets of reachable markings?

- **Trace Equivalence**: Given two marked nets, \((N_1, m_1)\) and \((N_2, m_2)\), with equal (or isomorphic) sets of transitions, do they have equal sets of valid firing sequences?

- **Language Equivalence**: Trace equivalence under mapping of transitions to a common alphabet

- **Bisimulation**: Equivalence under a bijection between markings

*In general, equivalence problems are undecidable*
A **structural property** is independent of initial marking $m_0$.

- **Structural Liveness:** Is there a marking $m$ such that $(N, m)$ is live?

- **Structural Boundedness:** Is $(N, m)$ bounded for every finite initial marking $m$?

- **Repetitiveness:** Is there a marking $m$ and a firing sequence $\sigma$ valid at $m$ such that a given transition (set of transitions) appears infinitely often in $\sigma$?

Deciding structural properties can be easier than corresponding problem for marked net.

**Structural boundedness and repetitiveness are in NP**
Bounded Petri nets are expressively equivalent to propositional STRIPS/PDDL
- Reachability is PSPACE-complete for both
- **Recall:** direct STRIPS to PN translation may blow up exponentially

General Petri nets are strictly more expressive than propositional STRIPS/PDDL

General Petri nets are at least as expressive as “lifted” (finite 1st order) STRIPS/PDDL
- probably also strictly more expressive (but no proof yet)
Counter TMs

- A $k$-counter machine ($k$CM) is a **deterministic** finite automaton with $k$ (positive) integer counters
  - can increment/decrement (by 1), or reset, counters
  - conditional jumps on $c_i > 0$ or $c_i = 0$

Note the differences:

- $k$CMs are **deterministic**: starting configuration determines unique execution; **Petri nets have choice**
- $k$CMs can **branch on** $c_i > 0/c_i = 0$; Petri nets can only precondition transitions on $m(p_i) > 0$

- A $k$CM is **$k$-bounded** iff no counter ever exceeds $k$
Counter TMs: Results

- An $n$-size TM can be simulated by an $O(n)$-size 2CM (if properly initialised)
  - Halting (i.e., reachability) for unbounded 2CMs is undecidable
  - PNs are strictly less expressive than unbounded 2CMs

- An $n$-size and $2^n$ space bounded TM can be simulated by $O(n)$-size $2^{2^n}$-bounded 2CM

- A $2^{2^n}$-bounded $n$-size 2CM can be (non-deterministically!) simulated by $O(n^2)$-size Petri net
  - Reachability for Petri nets is DSPACE($2^{\sqrt{n}}$)-hard
A vector \( y \in \mathbb{N}^{\mid P\mid} \) is **P-invariant** for \( N \) iff for any markings \( m[\ast]m' \), \( y^T m = y^T m' \)

**P-invariant** = linear combination of place markings that is invariant under any transition firing

A vector \( x \in \mathbb{N}^{\mid T\mid} \) is a **T-invariant** for \( N \) iff for any firing sequence \( \sigma \) such that \( n(\sigma) = x \) and any marking \( m \) where \( \sigma \) is enabled, \( m[\sigma]m \)

**T-invariant** = multiset of transitions whose combined effect is zero
Analysis Techniques
The Coverability Tree Construction

- The *coverability tree* of a marked net \((N, m_0)\) is an explicit representation of reachable markings – but not *exactly* the set of reachable markings.

- Constructed by forwards exploration:
  - Each enabled transition generates a successor marking.
  - If reach \(m\) such that \(m > m'\) for some ancestor \(m'\) of \(m\), replace \(m[i]\) by \(\omega\) for all \(i\) s.t. \(m[i] > m'[i]\).
  - \(m'[s = t_1, \ldots, t_l] m\), and since \(m \geq m'\), \(m[s] m''\) such that \(m'' \geq m\); sequence \(s\) can be repeated any number of times.
  - \(\omega\) means “arbitrarily large”.
  - Also check for regular loops (\(m = m'\) for some ancestor \(m'\) of \(m\)).
  - Every branch has finite depth.
Example

Introduction
Basics
Complexity & Expressivity
Analysis Techniques
Special Classes of Nets
Conclusion
Uses For The Coverability Tree

- Decides coverability:
  - \( m \) is coverable iff \( m \leq m' \) for some \( m' \) in the tree (where \( n < \omega \) for any \( n \in \mathbb{N} \)).
  - If \( m \) is coverable, there exists a covering sequence of length at most \( O(2^n) \).

- Decides boundedness:
  - \( (N, m_0) \) is unbounded iff there exists a self-covering sequence: \( m_0 [\sigma] m [\sigma'] m' \) such that \( m' > m \).
  - I.e., \( (N, m_0) \) is unbounded iff \( \omega \) appears in some marking in the coverability tree.
  - If \( (N, m_0) \) is unbounded, there exists a self-covering sequence of length at most \( O(2^n) \).

- In general, does not decide reachability.
  - Except if \( (N, m_0) \) is bounded.
The State Equation

- The **firing count vector** (a.k.a. **Parikh vector**) of a firing sequence $\sigma = t_{i_1}, \ldots, t_{i_l}$ is a $|T|$-dimensional vector $n(\sigma) = (n_1, \ldots, n_{|T|})$ where $n_i \in \mathbb{N}$ is the number of occurrences of transition $t_i$ in $\sigma$.

- If $m_0 \ll [\sigma] m'$, then

$$m' = m_0 + w(t_{i_1}) + \ldots + w(t_{i_l}) = m_0 + \sum_{j=1}^{\ldots|T|} w(t_j)n(\sigma)[j],$$

i.e., $m' = m_0 + Wn(\sigma)$.

- $m$ is reachable from $m_0$ only if $Wn = (m - m_0)$ has a solution $n \in \mathbb{N}^{|T|}$.

- This is a **necessary condition** but not sufficient.

- A solution $n$ is **realisable** iff, in addition, $n = n(\sigma)$ for some valid firing sequence $\sigma$. 
The State Equation & Invariance

- $y \in \mathbb{N}^{|P|}$ is a $P$-invariant iff it is a solution to $y^T W = 0$.
- $y^T m = y^T m_0$ for any $m$ reachable from $m_0$.
- $x \in \mathbb{N}^{|T|}$ is a $T$-invariant iff it is a solution to $W x = 0$.
- $m \langle \sigma \rangle m$ whenever $n(\sigma) = x$ and $\sigma$ enabled at $m$.
- Any (positive) linear combination of $P$-/T-invariants is a $P$-/T-invariant.
- The *reverse dual* of a net $N$ is obtained by swapping places for transitions and vice versa, and reversing all arcs.
- The incidence matrix of the reverse dual is the transpose of the incidence matrix of $N$.
- A $P$-(T-)invariant of $N$ is a T-(P-)invariant of the reverse dual.
Example: P-Invariants

\[
\begin{pmatrix}
1 \\
1 \\
0 \\
0 \\
1 \\
1 \\
0
\end{pmatrix}^T \begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & -1 \\
-1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 \\
0 \\
1 \\
1 \\
0 \\
1 \\
2
\end{pmatrix}^T \begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & -1 \\
-1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 \\
1 \\
0 \\
0 \\
1 \\
1 \\
0 \\
1 \\
2
\end{pmatrix}^T \begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & -1 \\
-1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]
Example: T-Invariants

\[
\begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
1 \\
1 \\
0 \\
0 \\
0 \\
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
0 \\
0 \\
1 \\
1 \\
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}
\]
Minimal Invariants

- The *support* of a P-/T-invariant $y$ is the set $\{i \mid y[i] > 0\}$. An invariant has *minimal support* iff no invariants support is a strict subset.
- The number of minimal support P-/T-invariants of a net is finite, but may be exponential.
- All P-/T-invariants are (positive) linear combinations of minimal support P-/T-invariants.
- A P-/T-invariant $y$ is *minimal* iff no $y' < y$ is invariant.
- A minimal invariant need not have minimal support.
- For each minimal support, there is a unique minimal invariant.
- Algorithms exist to generate all minimal support P-/T-invariants of a net.
The Fourier-Motzkin Algorithm for P-Invariants


2. For $j = 1, \ldots, |T|
   
   1. Append to $B$ all rows resulting from positive linear combinations of pairs of rows in $B$ that eliminate column $j$.
   2. Remove from $B$ all rows with non-zero $j$th element.

3. $B = [0 : D]$, where the rows of $D$ are P-invariants.
Example

\[ B = \begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \]
Example

\[
B = \begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]
Example

\[
B = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & -1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
Example

\[ B = \begin{pmatrix} 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 2 & 1 & 1 & 1 & 0 & 1 & 0 \\ \end{pmatrix} \]
Example

\[ B = \begin{pmatrix} 
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 2 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 2 & 1 
\end{pmatrix} \]
Example

\[
B = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

P-invariants:

\[z_1 = (1001001)\]
\[z_2 = (0010011)\]
\[z_3 = (1100110)\]
\[z_4 = (1110121)\]
The State Equation & Structural Properties

- $N$ is structurally bounded iff $y^T W \leq 0$ has a solution $y \in \mathbb{N}^{|P|}$ such that $y[i] \geq 1$ for $i = 1, \ldots, |P|$.
- $y$ is a linear combination of all place markings that is invariant or decreasing under any transition firing.
- $N$ is repetitive w.r.t. transition $t$ iff $Wx \geq 0$ has a solution $x \in \mathbb{N}^{|T|}$ such that $x[t] > 0$.
- $x$ is a multiset of transitions, including $t$ at least once, whose combined effect is zero or increasing.
- Can always find some initial marking $m_0$ from which $x$ is realisable.
Example

$z_1 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$ and
$z_4 = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 2 & 1 \end{pmatrix}$ are $P$-invariants of the net.

$y = z_1 + z_4 = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 2 & 2 \end{pmatrix}$ is also a $P$-invariant.

$y^TW = 0$ and $y \geq 1$: The net is **structurally bounded**.
Decidability of the (exact) reachability problem for general Petri nets was open for some time.

- Algorithm proposed by Sacerdote & Tenney in 1977 incorrect (or gaps in correctness proof).
- Correct algorithm by Mayr in 1981.
- Simpler correctness proof (for essentially the same algorithm) by Kosaraju in 1982.

Other algorithms have been presented since.

- All existing algorithms have unbounded complexity.
- Fun fact: A 2-EXP algorithm was proposed in 1998, but later shown to be incorrect.
Reachability: Preliminaries

- \( m \) is *semi-reachable* from \( m_0 \) iff there is a transition sequence \( s = t_{i_1}, \ldots, t_{i_n} \) such that
  \[
  m = m_0 + w(t_{i_1}) + \ldots + w(t_{i_n}).
  \]

- \( s \) is does not have to be valid (firable) at \( m_0 \).

- \( m \) is semi-reachable from \( m_0 \) iff \( Wn = (m - m_0) \) has a solution \( n \in \mathbb{N}^{|T|} \).

- If \( m \) is semi-reachable from \( m_0 \), then \( m + a \) is reachable from \( m_0 + a \) for some sufficiently large \( a \geq 0 \).
A controlled net is a pair of a marked net 
\( N = \langle P, T, F \rangle, m_0 \) and an NFA \( (A, q_0) \) over alphabet \( T \).

A defines a (regular) subset of (not necessarily firable) transition sequences.

Define reachability/coverability/boundedness for \( (N, m_0) \) w.r.t. \( A \) in the obvious way.

The coverability tree construction is easily modified to consider only sequences accepted by \( A \).

The reverse of \( N \), \( N_{\text{Rev}} \) (w.r.t. \( A \)) is obtained by reversing the flow relation (and arcs in \( A \)).

\[ W(N_{\text{Rev}}) = -W(N). \]
Reachability: A Sufficient Condition

In \((N, m_0)\) w.r.t. \((A, q_0)\), if

(a) \((m_*, q_*)\) is semi-reachable from \((m_0, q_0)\),
(b) \((m_0 + a, q_0)\) is reachable from \((m_0, q_0)\), for \(a \geq 1\),
(c) \((m_* + b, q_*)\) is reachable from \((m_*, q_*)\) in \(N_{Rev}\) w.r.t. \(A\), for \(b \geq 1\),
(d) \((b - a, q_*)\) is semi-reachable from \((0, q_*)\),

then \((m_*, q_*)\) is reachable from \((m_0, q_0)\).

The conditions above are effectively checkable:

(b) & (c) by coverability tree construction,
(a) & (d) through the state equation.
\[(m_0 + ka, q_0)\]

\[+a\]

\[(m_\ast + ka, q_\ast)\]

\[b - a\]

\[(m_\ast + kb, q_\ast)\]

\[-b\]

\[(m_\ast, q_\ast)\]
Consider a controlled net \((N, A)\) of the form,

\[
\begin{align*}
A_1 & \xrightarrow{t_{i_1}} A_2 \\
q_0 & \rightarrow m_0^{\text{out}} \rightarrow m_1^{\text{in}} \rightarrow m_2^{\text{out}} \rightarrow \cdots \rightarrow m_k^{\text{in}} \rightarrow q_*
\end{align*}
\]

with constraints \(m_{i}^{\text{in/ out}}[j] = x_{i,j}^{i/o}\) or \(m_{i}^{\text{in/ out}}[j] \geq y_{i,j}^{i/o} \geq 0\).

1. If the sufficient reachability condition holds for each \((m_{i}^{\text{in}}, q_{i}^{\text{in}})\) and \((m_{i}^{\text{out}}, q_{i}^{\text{out}})\) w.r.t \(A_i\), then \((m_{*}, q_{*})\) is reachable from \((m_{0}, q_{0})\).

2. Let \(\Delta(A_i) = \{m | m = Wn(s), s \in L(A_i)\}\).

3. Let \(\Gamma = \{m_{i}^{\text{in}}, m_{i}^{\text{out}}, n_i | m_{i+1}^{\text{in}} - m_{i}^{\text{out}} = w(t_{i}), m_{i}^{\text{out}} - m_{i}^{\text{in}} \in \Delta(A_i), \text{and constraints hold}\}\).

4. If \((m_{0}, q_{0}) [s] (m_{*}, q_{*}), s\) defines an element in \(\Gamma\).
\( \Gamma \) is a \textit{semi-linear set}: consistency (non-emptiness) is decidable via Pressburger arithmetic.

If \( \Gamma \) is consistent, but the sufficient condition does not hold in some \( A_i \), then \( A_i \) can be replaced by a new “chain” of controllers, \( A^1_i, \ldots, A^l_i \), each of which is “simpler”:

- more equality constraints (\( m^\text{in/out}_{i,l} = x_{i,l,j} \)), or
- same equality constraints and smaller automaton.

There can be several possible replacements (non-deterministic choice).

If \( (m^*_*, q^*_*) \) is not reachable from \( (m_0, q_0) \), every choice (branch) eventually leads to an inconsistent system.
Special Classes of Nets
Special Classes of Nets

- **State Machines:**
  - every transition has one incoming and one outgoing arc
  - i.e. $|\cdot t| = |t\cdot| = 1$ for each $t \in T$.

- **Marked Graphs:**
  - every place has one incoming arc, and one outgoing arc
  - i.e. $|\cdot p| = |p\cdot| = 1$ for each $p \in P$.

- **Free-choice Nets:**
  - every arc is either the only arc going from the place, or only arc going to the transition
  - i.e. $|p\cdot| \leq 1$ or $\cdot(p\cdot) = \{p\}$ for each $p \in P$. 
Marked Graphs

- An ordinary Petri net with $|\bullet p| = |p\bullet| = 1$ for each place $p$ is a $T$-graph, or marked graph.
- Abstracting away places leaves a directed graph:
  - Called the underlying graph (usually denoted $G$).
  - A marking of the net is a marking of the edges of $G$.
- Marked graphs model “decision-free” concurrent systems.
- Several properties of marked graphs are decidable in polynomial time:
  - Structural liveness and boundedness.
  - Liveness and boundedness for a given initial marking.
  - Simple condition for realisability (and thus reachability).
Example: Marked and Underlying Graphs
Some Properties of Marked Graphs

- **Theorem:** The total number of tokens on every directed circuit in the underlying graph is invariant.

- **Theorem:** The maximum number of tokens an edge \( a \rightarrow b \) in \( (G, m_0) \) can ever have is equal to the minimum number of tokens \( m_0 \) places on any directed circuit that contains this edge.

- **Theorem:** A marked graph \( (G, m_0) \) is live iff \( m_0 \) places at least one token on every directed circuit of \( G \).

- **Theorem:** A live marked graph \( (G, m_0) \) is \( k \)-bounded iff every place (edge in \( G \)) belongs to a directed circuit and \( m_0 \) places at most \( k \) tokens on every directed circuit of \( G \).

- **Theorem:** A marked graph net has a live and bounded marking iff \( G \) is strongly connected.
Free Choice Nets

- An ordinary Petri net such that $|p^\bullet| \leq 1$ or $\bullet(p^\bullet) = \{p\}$ for each place $p$, is a free choice net.
- Equivalently: If $p^\bullet \cap p'^\bullet \neq \emptyset$ then $|p^\bullet| = |p'^\bullet| = 1$, for all $p, p' \in P$.
- Extended free choice net: If $p^\bullet \cap p'^\bullet \neq \emptyset$ then $p^\bullet = p'^\bullet$, for all $p, p' \in P$.
- An extended free choice net can be transformed to a basic free choice net, adding at most a linear number of places and transitions.
- Note: Marked graphs and state machines are also free choice nets.
- A fundamental property of free choice nets: if $\bullet t \cap \bullet t' \neq \emptyset$ then whenever $t$ is enabled, so is $t'$.
Example: Free Choice Nets

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Example: Free Choice Nets

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Example: Free Choice Nets
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Decomposition of Free Choice Nets

- A subnet of $N = (P, T, W)$ is a net $N' = (P', T', W')$ with $P' \subseteq P$, $T' \subseteq T$ and $W' = W|_{(P' \cup T')}$.

- The P-subnet induced by $S \subseteq P$ is $(S, \bullet S \cup S^\bullet, W')$.
  - That is, the subnet consisting of $S$ and all transition incident on a place in $S$.

- The T-subnet induced by $U \subseteq T$ is $(\bullet U \cup U^\bullet, U, W')$.
  - That is, the subnet consisting of $U$ and all places incident on a transition in $U$.

- A **P-component** is a strongly connected P-subnet such that $|\bullet t|, |t^\bullet| \leq 1$, for all $t$.
  - A P-component is a state machine.

- A **T-component** is a strongly connected T-subnet such that $|\bullet p|, |p^\bullet| \leq 1$, for all $p$.
  - A T-component is a marked graph.
Decomposition of Free Choice Nets

- **Theorem**: A live free choice net \((N, m_0)\) is 1-bounded (safe) iff it is covered by P-components, each of which has a single token at \(m_0\).

- **Theorem**: A live and safe free choice net \((N, m_0)\) is covered by T-components, and for each T-component, \(N'\), there is a reachable marking \(m\) such that \((N', m|_{N'})\) is live and safe.
Example: Decomposition into State Machines
Example: Decomposition into Marked Graphs
Siphons and Traps

- A **siphon** is a subset \( S \) of places such that \( \bullet S \subseteq S^\bullet \).
  - Every transition that outputs a token to a place in \( S \) also consumes a token from a place in \( S \).
  - If \( m \) places no token in \( S \), no marking reachable from \( m \) does either.

- A **trap** is a subset \( S \) of places such that \( S^\bullet \subseteq \bullet S \).
  - Every transition that consumes a token from a place in \( S \) also outputs a token to a place in \( S \).
  - If \( m \) places at least one token in \( S \), so does every marking reachable from \( m \).

**Theorem:** A free choice net \((N, m_0)\) is live iff every siphon contains a marked trap.
The only siphon is $P$.

$P$ is also a trap.

$\{p_2\}$ and $\{p_3, p_4\}$ are siphons.

$\{p_1\}$ and $\{p_3, p_4\}$ are traps.
Some Complexity Results for Free Choice Nets

- Liveness for marked free choice nets is decidable in polynomial time.
- Boundedness of *live* free choice nets is decidable in polynomial time.
- A number of properties of *live and bounded* free choice nets are decidable in polynomial time, e.g.,
  - Transition executability and repeated executability.
  - The “home state” property (markings that can always be re-reached).
- Reachability in free choice nets is NP-hard.
Characterisation by Derivation Rules

- Initial net: \( \text{\includegraphics[width=1cm]{initial_net}} \)

- Rule #1: Add a new place \( p' \) with \( r(p') = \sum_{p \in P} \lambda_p r(p) \) and \( |p'\cdot| = 1 \).

- Rule #2: Replace place \( p \) with a connected P-graph \( N' \), and connect each input and output of \( p \) to at least one place in \( N' \).
  - Must have \( |\cdot p| > 1 \) and \( |p\cdot| > 1 \), except for initial net.
  - Every place \( p' \in N' \) must appear on a path in the resulting net that enters and leaves \( N' \).

**Theorem:** The class of nets obtained by applying the above rules to the initial net is exactly the class of structurally live and structurally bounded free choice nets.
Reachability: Acyclic Nets

- Recall: $m_0 [s] m$ implies $\exists n \in \mathbb{N}^{|T|} : Wn = (m - m_0)$.

- A solution $n$ is realisable iff $n = n(s)$ for some valid firing sequence $s$.

- **Theorem:** For an acyclic net, every solution to $Wn = (m - m_0)$ is realisable.

- Reachability in acyclic nets is NP-hard.
**Theorem:** In a live marked graph, \( m \) is reachable from \( m_0 \) iff \( m_0 \) and \( m \) place the same total number of tokens on every fundamental circuit of the underlying graph.

- A fundamental circuit is obtained by adding one edge to a spanning tree.
- The directed fundamental circuits of a marked graph are a full set of linearly independent P-invariants.
Conclusions
Petri nets: Intuitive, graphical modelling formalism, closely related to planning.

Petri net theory offers a different set of tools:
- Algebraic methods (based on the state equation).
- Characterisation and study of classes of nets with special structure.

Planning also has tools potentially applicable to Petri nets.
The Many Things We Haven’t Talked About

- Extensions of basic Place-Transition nets:
  - Read arcs, reset arcs and inhibitor arcs.
  - Colored Petri nets, timed nets, stochastic nets, etc.

- Other properties of Petri nets (and related decision problems):
  - Model checking (tense logics, process calculi).
  - Language (trace) properties.

- Heaps more results concerning different Petri net subclasses.