Petri Nets (for Planners)

B. Bonet, P. Haslum

... from various places ...

ICAPS 2011

Introduction & Motivation

- Petri Nets (PNs) is formalism for modelling discrete event systems
- Developed by (and named after) C.A. Petri in 1960s
- In general Petri nets, places are **unbounded counters**
  - advantages in expressivity and modelling convenience
  - questions of reachability, coverability, etc. are computationally harder to answer, but still decidable

Exchange of ideas between Petri nets and planning holds potential to benefit both areas:

- Analysis methods for Petri nets are often based on ideas & techniques not common in planning:
  - algebraic methods based on the state equation
  - rich literature on the study of classes of nets with special structure
- Yet, some standard planning techniques (e.g., search heuristics) are unknown in the PN community

Outline of the Tutorial

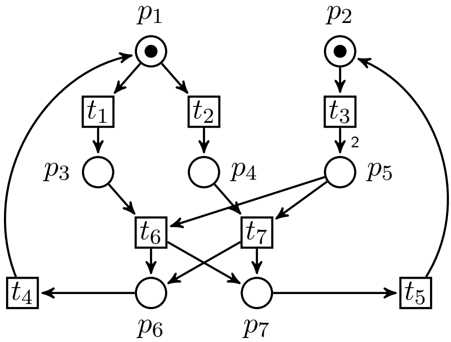
1. Definitions, notation and modelling
2. Decision problems, complexity and expressivity
3. Analysis techniques for general Petri nets
   - Coverability
   - The state equation
   - Reachability
4. Petri nets with special structure
5. Conclusions

Definitions, Notation and Modelling

Terminology and Intuition

- A Petri net has **places**, **transitions**, and **directed arcs**
- Arcs connect places to transitions or vice versa
- Places contain zero or finite number of **tokens**
- A **marking** is disposition of tokens in places
- A transition is **fireable** if there is token at the start place of **each input arc**
- When transition fires:
  - it **consumes** token from start place of each input arc
  - it **puts** token at end place of each output arc
- Execution is **non-deterministic**

Example



Formal Definition

Place/Transition (P/T) net is tuple $N = (P, T, W)$ where:

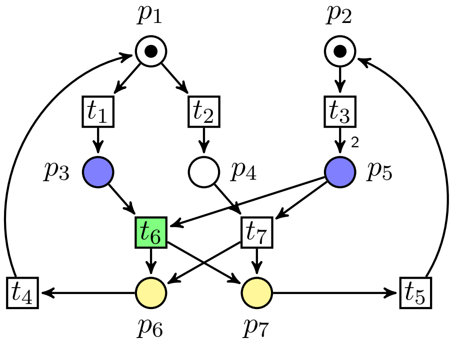
- P is set of places
- T is set of transitions (and $P \cap T = \emptyset$)
- $W \subseteq (P \times T) \cup (T \times P) \to \mathbb{N}$
  (**multiset** of arcs: each (x, y) has **multiplicity** $W(x, y)$)

For transition t:

- **preset** is ${}^\bullet t = \{s : W(s, t) > 0\}$ (input places)
- **postset** is $t^\bullet = \{s : W(t, s) > 0\}$ (output places)

Marking is $\mathbf{m} : P \to \mathbb{N}$ (zero or more tokens at each place)

Example



- marking $\mathbf{m} = \langle 1, 1, 0, 0, 0, 0, 0 \rangle$
- transition t_6: ${}^\bullet t_6 = \{p_3, p_5\}$, $t_6{}^\bullet = \{p_6, p_7\}$

Execution Semantics

A transition t is **enabled** or **firable** at marking $\mathbf{m}$ if

$$\mathbf{m}(p) \geq W(p,t) \quad \text{for each } p \in {}^{\bullet}t$$

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Upon firing, t **produces** new marking $\mathbf{m}'$ such that

$$\mathbf{m}'(p) = \begin{cases} \mathbf{m}(p) - \overbrace{W(p,t)}^{\text{consumed}} + \overbrace{W(t,p)}^{\text{added}} & \text{if } p \in {}^\bullet t \cup t^\bullet \\ \mathbf{m}(p) & \text{if } p \notin {}^\bullet t \cup t^\bullet \end{cases}$$

Execution Semantics

Transition relations:

- $\mathbf{m}\,[t\rangle\,\mathbf{m}'$ if t is enabled at $\mathbf{m}$ and produces $\mathbf{m}'$
- $\mathbf{m}\,[\sigma\rangle\,\mathbf{m}'$, for sequence $\sigma = t_1 t_2 \cdots t_n$, if exists $\mathbf{m}''$ with
  - $\mathbf{m}\,[t_1\rangle\,\mathbf{m}''$
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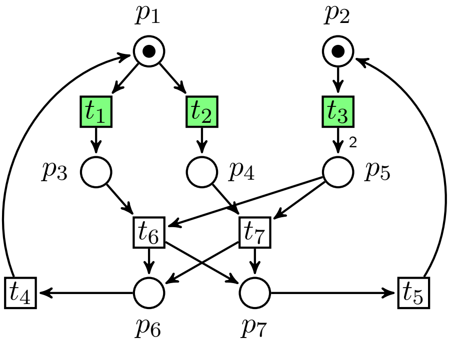
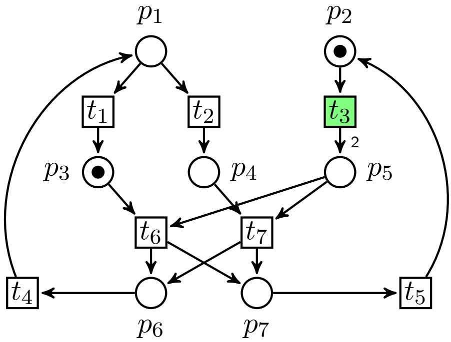
Firing sequences $L(N, \mathbf{m}_0) = \{\sigma \in T^{<\infty} : \exists \mathbf{m}.\mathbf{m}_0\,[\sigma\rangle\,\mathbf{m}\}$

Example



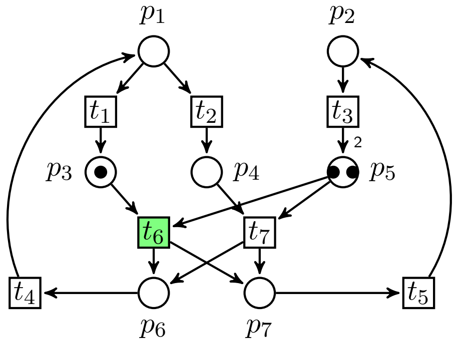
- **marking** $\mathbf{m} = \langle 1, 1, 0, 0, 0, 0, 0 \rangle$
- **enabled:** t_1, t_2, t_3

Example



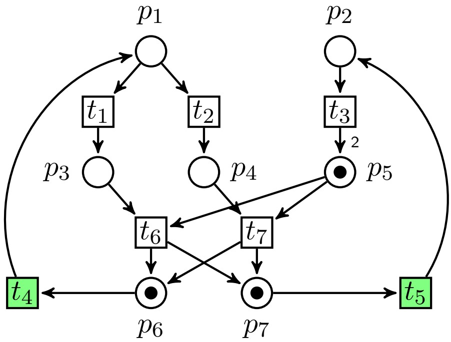
- **marking** $\mathbf{m} = \langle 0, 1, 1, 0, 0, 0, 0 \rangle$
- **enabled:** t_3

Example



- **marking** $\mathbf{m} = \langle 0, 0, 1, 0, 2, 0, 0 \rangle$
- **enabled:** t_6

Example



- **marking** $\mathbf{m} = \langle 0, 0, 0, 0, 1, 1, 1 \rangle$
- **enabled:** t_4, t_5

Arithmetic of Functions

For two functions $f, g \in \mathbb{N}^X$:

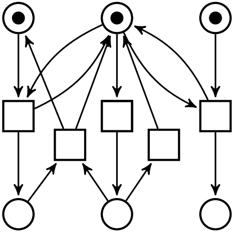
- $f \geq g$ if $f(x) \geq g(x)$ for each place x
- $f > g$ if $f \geq g$ and there is x such that $f(x) > g(x)$
- $f + g$ defined pointwise as $(f+g)(x) = f(x) + g(x)$

Hence, $\mathbf{m}\,[t\rangle\,\mathbf{m}'$ iff

$$\mathbf{m} \geq W(\cdot, t) \qquad \text{(enable condition)}$$
$$\mathbf{m}' = \mathbf{m} - W(\cdot, t) + W(t, \cdot)$$

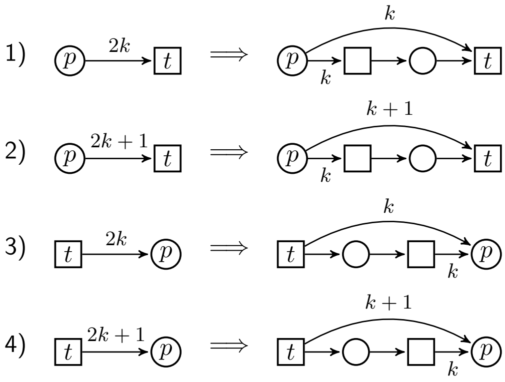
Ordinary Nets

A P/T net $N = (P, T, W)$ is **ordinary** iff $W(p, t) \leq 1$ for all p, t



Thm: any net can be transformed into **equivalent** ordinary net

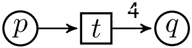
Transformation Rules



Each rule **decrease multiplicity** by half and add 2 nodes

Resulting size is $O(\sum_{x,y} W(x,y))$ (exponential)

Example



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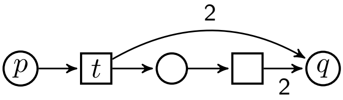

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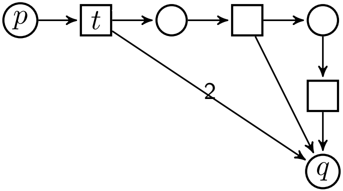

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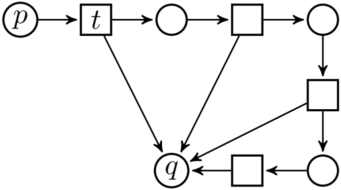


Types of Nets

- Marking $\mathbf{m}$ is **k-bounded** if $\mathbf{m}(p) \leq k$ for all $p \in P$
- Marked net $(N, \mathbf{m}_0)$ is **k-bounded** if every reachable marking is k-bounded
- It is **bounded** if it is k-bounded for some k
- It is **safe** if it is 1-bounded

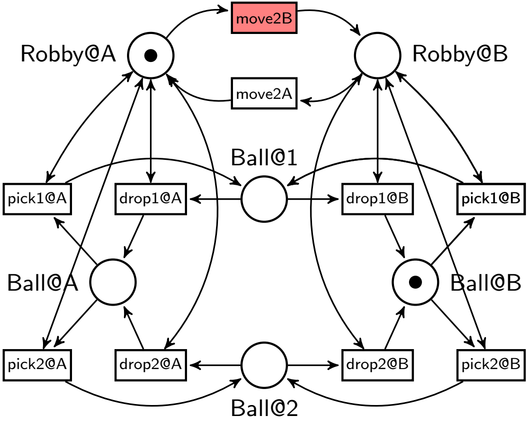
Safe Networks

- Every reachable marking is 1-bounded
- Marking $\mathbf{m}$ can be thought as **state** where places represents fluents:
  - if $\mathbf{m}(p) = 1$ then fluent p is **true** at $\mathbf{m}$
  - if $\mathbf{m}(p) = 0$ then fluent p is **false** at $\mathbf{m}$
- Safe networks can be used for STRIPS planning

Direct STRIPS to PN Translations

- Each atom is a place
- Each (grounded) action is a transition t:
  - input arcs $p \to t$ for each precondition p
  - output arcs $t \to p$ for each positive effect p
  - output arcs $t \to p$ for each precondition p that is not deleted nor added
- Initial state gives initial marking
- Goal state gives **partial** desired marking
- Plan existence becomes "Coverability" problem

Example: Gripper w/ 1 Ball and 2 Arms



move2B, pick1@B, move2A, drop1@A

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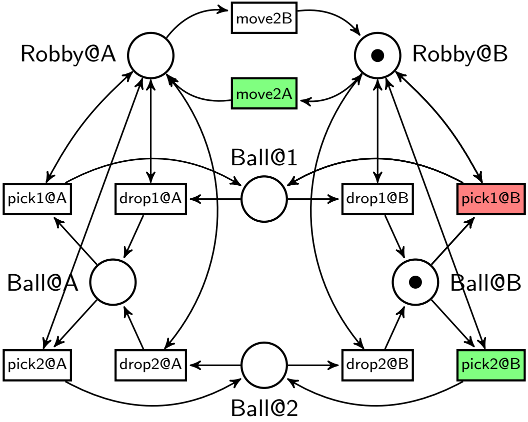

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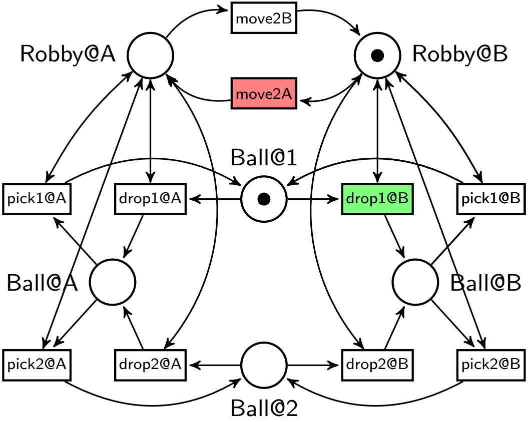

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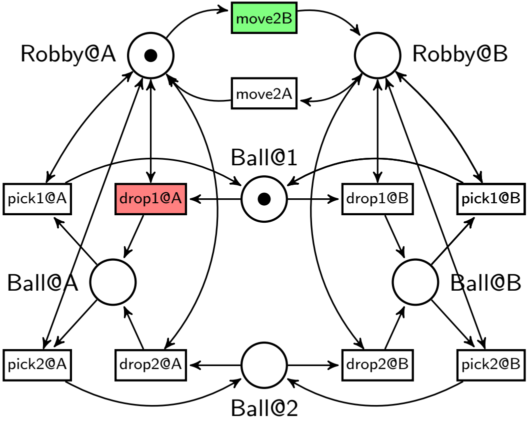

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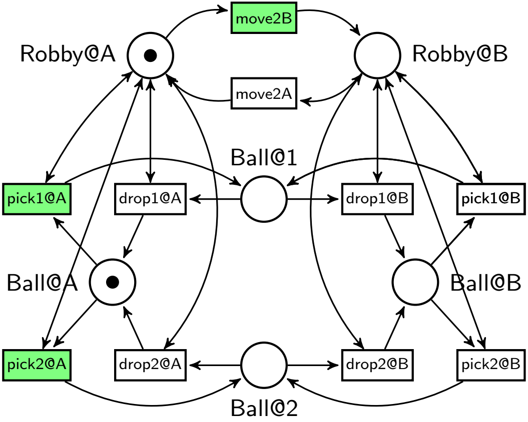

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Safe STRIPS Problems

STRIPS problem is **safe** if its direct translation $(N, \mathbf{m}_0)$ is **safe**

Sufficient Condition:

- For each added atom p, there is precondition q that is deleted such that $\{p, q\}$ is mutex

Enforcing the condition:

- Add 'not-p' atoms for each atom p
- For each action that contains a deleted atom p that is not precondition, generate two similar actions with p and not-p in precondition (respectively)
- Worst-case size of pre-processing is exponential in number of atoms that are deleted and don't appear as preconditions

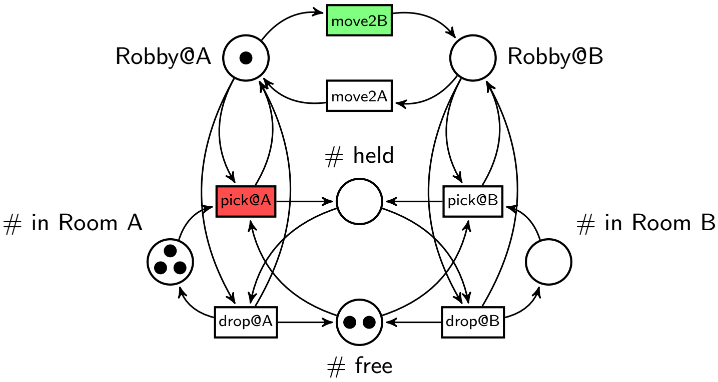
Modelling Planning Problems

General nets can “store” multiple tokens at single place

Places can be used to represent:

- **number** of identical objects at location
- **resource quantity**

Gripper with 3 identical balls and 2 identical arms



pick@A, pick@A, move2B, drop@B, drop@B, move2A, pick@A, move2B, drop@B

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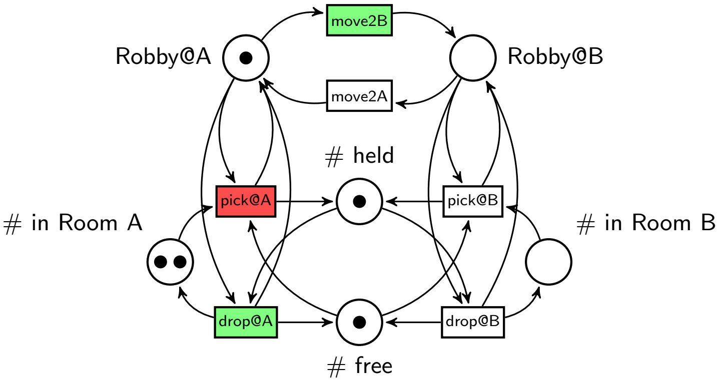

pick@A, **pick@A**, move2B, drop@B, drop@B, move2A, pick@A, move2B, drop@B

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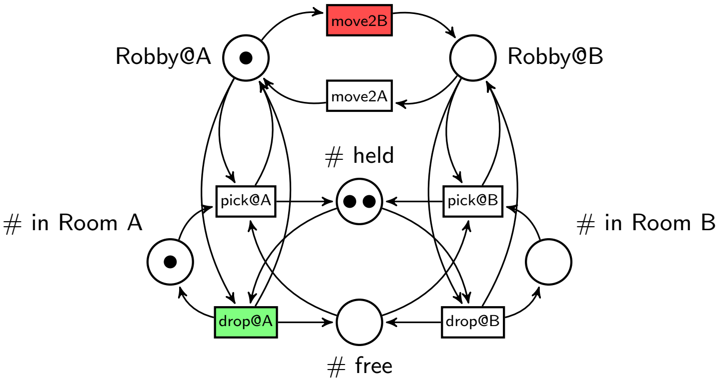

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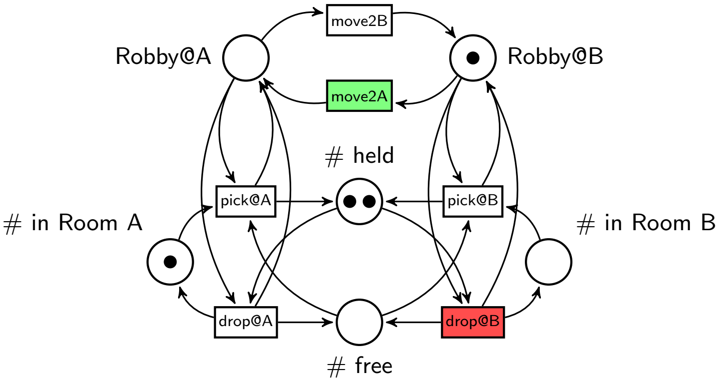

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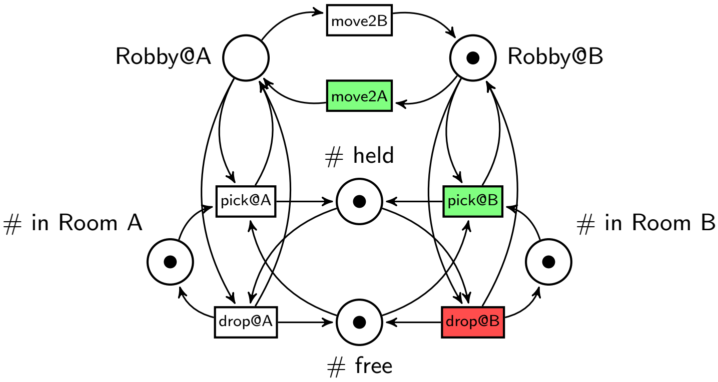

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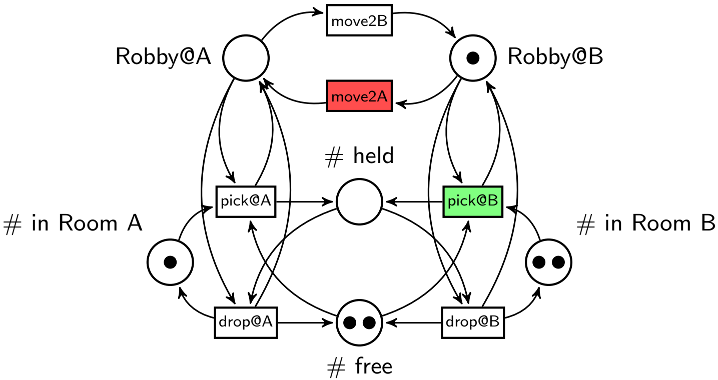

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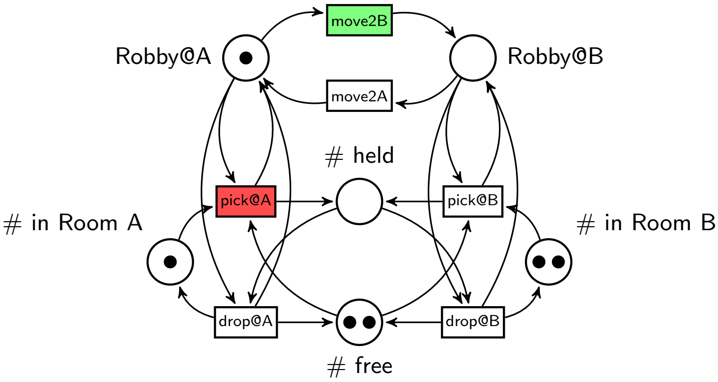

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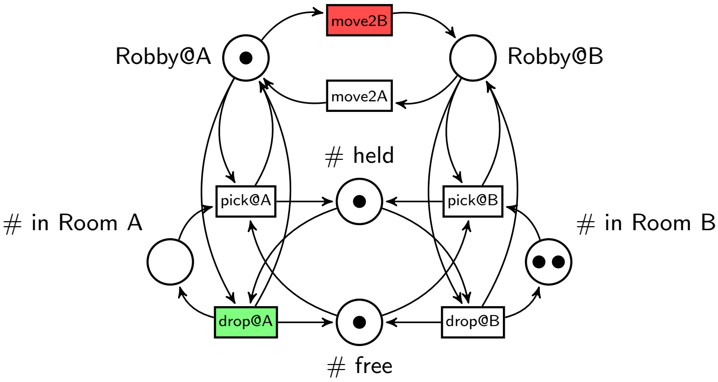

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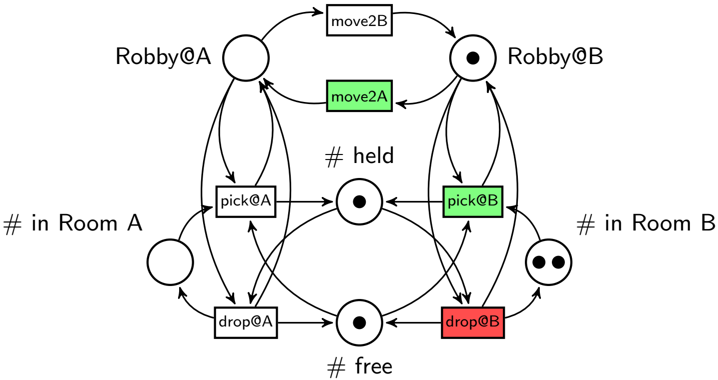

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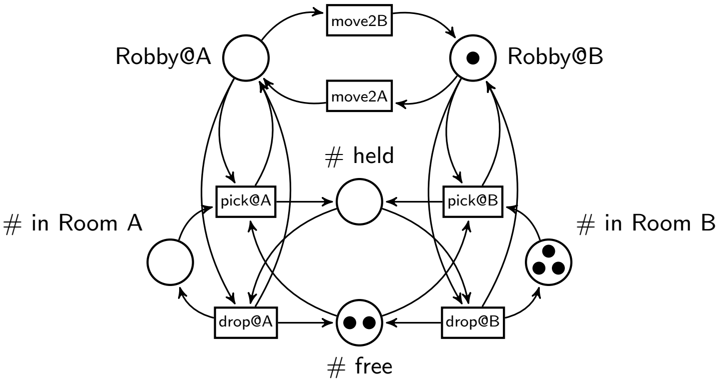

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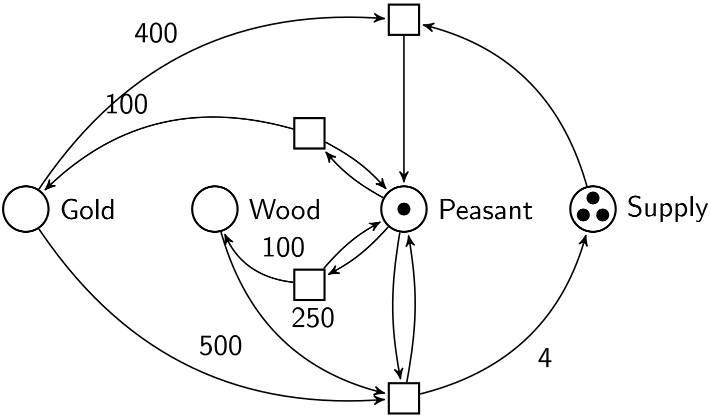
Wargus Domain (Chan et al. 2007)



Other Types of Nets

State Machines:

- every **transition** has one incoming and one outgoing arc
  i.e. $|^\bullet t| = |t^\bullet| = 1$ for each $t \in T$

Marked Graphs:

- every **place** has one incoming arc, and one outgoing arc
  i.e. $|^\bullet p| = |p^\bullet| = 1$ for each $p \in P$

Free-choice Nets:

- every arc is either the **only arc going from** the place, or **only arc going to** the transition
  i.e. $|p^\bullet| \leq 1$ or $^\bullet(p^\bullet) = \{p\}$ for each $p \in P$

Extensions

Inhibitor arcs (enablers):

- transition enabled when there is **no token at place**

Read arcs (enablers):

- do not **consume** tokens

Reset arcs: erase all tokens at place

Others:

- colored, hierarchical, prioritization, ...

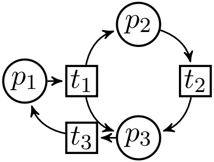
Vector Notation

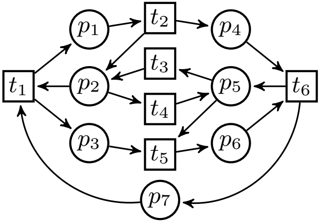
Two **vectors** associated with transition t:

$$\mathbf{W}_t^- = \begin{pmatrix} W(p_1, t) \\ \vdots \\ W(p_{|P|}, t) \end{pmatrix} \qquad \mathbf{W}_t^+ = \begin{pmatrix} W(t, p_1) \\ \vdots \\ W(t, p_{|P|}) \end{pmatrix}$$

- t enabled at $\mathbf{m}$ iff $\mathbf{m} \geq \mathbf{W}_t^-$
- $\mathbf{W}_t = \mathbf{W}_t^+ - \mathbf{W}_t^-$ is **effect** of t
- firing t leads to $\mathbf{m}' = \mathbf{m} + \mathbf{W}_t$
- $\mathbf{W} = \left(\mathbf{W}_{t_1}, \mathbf{W}_{t_2}, \ldots, \mathbf{W}_{t_{|T|}}\right)$ is **incidence matrix**
- $\mathbf{r}_p$: row of $\mathbf{W}$ corresponding to place p

Examples



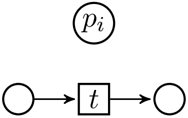


$$\mathbf{W} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

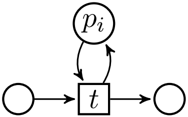
$$\mathbf{W} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Representation Ambiguity and Pure Nets

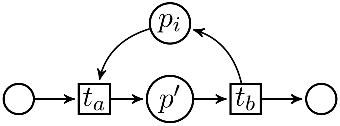
$\mathbf{W}_t[i] = 0$:



or



- **Pure nets** have no "self loops": ${}^\bullet t \cap t^\bullet = \emptyset$ for every transition t
- For pure nets, incidence matrix $\mathbf{W}$ **unambiguously** defines the net
- Any net can be transformed into a pure net by **splitting loops**:



Transformation is **linear space**

Complexity & Expressivity

Decision Problems for Marked Nets

Given a marked net $(N, \mathbf{m}_0)$:

- **Reachability:** Is there a firing sequence that ends with given marking $\mathbf{m}$?
- **Coverability:** Is there a firing sequence that ends with marking $\mathbf{m}'$ such that $\mathbf{m}' \geq \mathbf{m}$ for given $\mathbf{m}$?
- **Boundedness:** Does there exist a integer k such that every reachable marking is k-bounded? $\mathbf{m} \leq \mathbf{K}$?

Coverability and boundedness are EXPSPACE-complete

Reachability is EXPSPACE-hard, but existing algorithms are non-primitive recursive (i.e., have unbounded complexity)

More Properties

- **Executability:** Is there a firing sequence valid at $\mathbf{m}_0$ that includes transition t?
  - Reduces to coverability: t is executable iff $\mathbf{W}_t^-$ is coverable
  - and vice versa: reduction using a "goal transition"
- **Repeated Executability:** Is there a firing sequence in which a given transition (or set of transitions) occurs an infinite number of times?
- **Reachable Deadlock:** Is there a reachable marking $\mathbf{m}$ at which no transition is enabled?
- **Liveness:** Executability of every transition at every reachable marking, i.e., $\forall M : M_0 [*\rangle M \to \forall t \, \exists M', M'' : M [*\rangle M' [t\rangle M''$.

...and many more ...

Equivalence Problems

- **Equivalence:** Given two marked nets, $(N_1, \mathbf{m}_1)$ and $(N_2, \mathbf{m}_2)$, with equal (or isomorphic) sets of places, do they have the equal sets of reachable markings?
- **Trace Equivalence:** Given two marked nets, $(N_1, \mathbf{m}_1)$ and $(N_2, \mathbf{m}_2)$, with equal (or isomorphic) sets of transitions, do they have equal sets of valid firing sequences?
- **Language Equivalence:** Trace equivalence under mapping of transitions to a common alphabet
- **Bisimulation:** Equivalence under a bijection between markings

In general, equivalence problems are undecidable

Structural Properties

A **structural property** is **independent** of initial marking $\mathbf{m}_0$

- **Structural Liveness:** It there a marking $\mathbf{m}$ such that $(N, \mathbf{m})$ is live?
- **Structural Boundedness:** Is $(N, \mathbf{m})$ bounded for every finite initial marking $\mathbf{m}$?
- **Repetitiveness:** Is there a marking $\mathbf{m}$ and a firing sequence σ valid at $\mathbf{m}$ such that a given transition (set of transitions) appears infinitely often in σ?

Deciding structural properties can be easier than corresponding problem for marked net

Structural boundedness and repetitiveness are in NP

Complexity: Implications Of and For Expressivity

- Bounded Petri nets are **expressively equivalent** to propositional STRIPS/PDDL
  - Reachability is PSPACE-complete for both
  - **Recall:** direct STRIPS to PN translation may blow up exponentially
- General Petri nets are **stictly more expressive** than propositional STRIPS/PDDL
- General Petri nets are **at least as expressive** as “lifted” (finite 1st order) STRIPS/PDDL
  - probably also strictly more expressive (but no proof yet)

Counter TMs

- A k-*counter machine* (kCM) is a **deterministic** finite automaton with k (positive) integer counters
  - can increment/decrement (by 1), or reset, counters
  - conditional jumps on $c_i > 0$ or $c_i = 0$
- Note the differences:
  - kCMs are **deterministic**: starting configuration determines unique execution; **Petri nets have choice**
  - kCMs can **branch on** $c_i > 0/c_i = 0$; Petri nets can only precondition transitions on $\mathbf{m}(p_i) > 0$
- A kCM is k-**bounded** iff no counter ever exceeds k

Counter TMs: Results

- An n-size TM can be simulated by an $O(n)$-size 2CM (if properly initialised)
  - Halting (i.e., reachability) for unbounded 2CMs is undecidable
  - PNs are strictly **less expressive** than unbounded 2CMs
- An n-size and 2^n space bounded TM can be simulated by $O(n)$-size 2^{2^n}-bounded 2CM
- A 2^{2^n}-bounded n-size 2CM can be (non-deterministically!) simulated by $O(n^2)$-size Petri net
  - **Reachability for Petri nets is DSPACE($2^{\sqrt{n}}$)-hard**

Invariants

- A vector $\mathbf{y} \in \mathbb{N}^{|P|}$ is **P-invariant** for N iff for any markings $\mathbf{m}\,[*\rangle\,\mathbf{m}'$, $\mathbf{y}^T\mathbf{m} = \mathbf{y}^T\mathbf{m}'$

  P-invariant = linear combination of place markings that is invariant under any transition firing

- A vector $\mathbf{x} \in \mathbb{N}^{|T|}$ is a **T-invariant** for N iff for any firing sequence σ such that $\mathbf{n}(\sigma) = \mathbf{x}$ and any marking $\mathbf{m}$ where σ is enabled, $\mathbf{m}\,[\sigma\rangle\,\mathbf{m}$

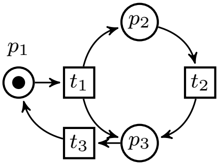
  T-invariant = multiset of transitions whose combined effect is zero

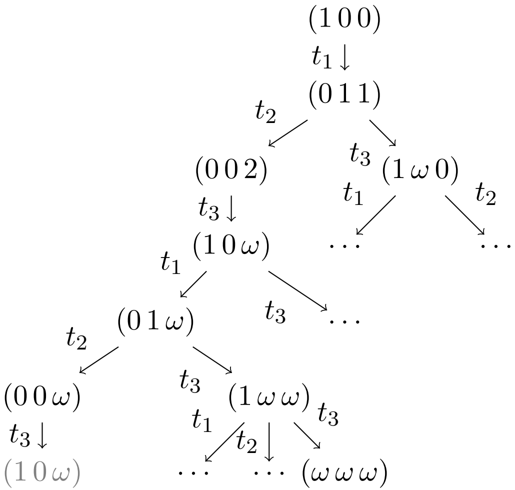
Analysis Techniques

The Coverability Tree Construction

- The *coverability tree* of a marked net $(N, \mathbf{m}_0)$ is an explicit representation of reachable markings – but not *exactly* the set of reachable markings.
- Constructed by forwards exploration:
  - Each enabled transition generates a successor marking.
  - If reach $\mathbf{m}$ such that $\mathbf{m} > \mathbf{m}'$ for some ancestor $\mathbf{m}'$ of $\mathbf{m}$, replace $\mathbf{m}[i]$ by ω for all i s.t. $\mathbf{m}[i] > \mathbf{m}'[i]$.
    - $\mathbf{m}' \, [s = t_1, \ldots, t_l\rangle \, \mathbf{m}$, and since $\mathbf{m} \geq \mathbf{m}'$, $\mathbf{m} \, [s\rangle \, \mathbf{m}''$ such that $\mathbf{m}'' \geq \mathbf{m}$; sequence s can be repeated any number of times.
    - ω means "arbitraribly large".
  - Also check for regular loops ($\mathbf{m} = \mathbf{m}'$ for some ancestor $\mathbf{m}'$ of $\mathbf{m}$).
- Every branch has finite depth.

Example


p_1
p_2
p_3
t_1
t_2
t_3


$(1\,0\,0)$
t_1
$(0\,1\,1)$
t_2
t_3
$(0\,0\,2)$
$(1\,\omega\,0)$
t_3
t_1
t_2
$(1\,0\,\omega)$
$\ldots$
$\ldots$
t_1
t_3
$(0\,1\,\omega)$
$\ldots$
t_2
t_3
$(0\,0\,\omega)$
$(1\,\omega\,\omega)$
t_3
t_1
t_2
t_3
$(1\,0\,\omega)$
$\ldots$
$\ldots$
$(\omega\,\omega\,\omega)$

Uses For The Coverability Tree

- Decides coverability:
  - $\mathbf{m}$ is coverable iff $\mathbf{m} \leq \mathbf{m}'$ for some $\mathbf{m}'$ in the tree (where $n < \omega$ for any $n \in \mathbb{N}$).
  - If $\mathbf{m}$ is coverable, there exists a covering sequence of length at most $O(2^n)$.
- Decides boundedness:
  - $(N, \mathbf{m}_0)$ is unbounded iff there exists a self-covering sequence: $\mathbf{m}_0 [\sigma\rangle \mathbf{m} [\sigma'\rangle \mathbf{m}'$ such that $\mathbf{m}' > \mathbf{m}$.
  - I.e., $(N, \mathbf{m}_0)$ is unbounded iff ω appears in some marking in the coverability tree.
  - If $(N, \mathbf{m}_0)$ is unbounded, there exists a self-covering sequence of length at most $O(2^n)$.
- In general, does *not* decide reachability.
  - Except if $(N, \mathbf{m}_0)$ is bounded.

The State Equation

- The *firing count vector* (a.k.a. *Parikh vector*) of a firing sequence $\sigma = t_{i_1}, \ldots, t_{i_l}$ is a $|T|$-dimensional vector $\mathbf{n}(\sigma) = (n_1, \ldots, n_{|T|})$ where $n_i \in \mathbb{N}$ is the number of occurrences of transition t_i in σ.
- If $\mathbf{m_0}\,[\sigma\rangle\,\mathbf{m}'$, then

  $$\mathbf{m}' = \mathbf{m}_0 + \mathbf{w}(t_{i_1}) + \ldots + \mathbf{w}(t_{i_l}) = \mathbf{m}_0 + \sum_{j=1\ldots|T|} \mathbf{w}(t_j)\mathbf{n}(\sigma)[j],$$

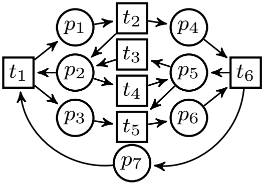
  i.e., $\mathbf{m}' = \mathbf{m_0} + W\mathbf{n}(\sigma)$.
- $\mathbf{m}$ is reachable from $\mathbf{m_0}$ only if $W\mathbf{n} = (\mathbf{m} - \mathbf{m_0})$ has a solution $\mathbf{n} \in \mathbb{N}^{|T|}$.
- This is a **necessary condition** but not sufficient.
  - A solution $\mathbf{n}$ is *realisable* iff, in addition, $\mathbf{n} = \mathbf{n}(\sigma)$ for some valid firing sequence σ.

The State Equation & Invariance

- $\mathbf{y} \in \mathbb{N}^{|P|}$ is a P-invariant iff it is a solution to $\mathbf{y}^T W = \mathbf{0}$.
  - $\mathbf{y}^T \mathbf{m} = \mathbf{y}^T \mathbf{m}_0$ for any $\mathbf{m}$ reachable from $\mathbf{m}_0$.
- $\mathbf{x} \in \mathbb{N}^{|T|}$ is a T-invariant iff it is a solution to $W\mathbf{x} = \mathbf{0}$.
  - $\mathbf{m} \left[\sigma\right\rangle \mathbf{m}$ whenever $\mathbf{n}(\sigma) = \mathbf{x}$ and σ enabled at $\mathbf{m}$.
- Any (positive) linear combination of P-/T-invariants is a P-/T-invariant.
- The *reverse dual* of a net N is obtained by swapping places for transitions and vice versa, and reversing all arcs.
  - The incidence matrix of the reverse dual is the transpose of the incidence matrix of N.
  - A P-(T-)invariant of N is a T-(P-)invariant of the reverse dual.

Example: P-Invariants

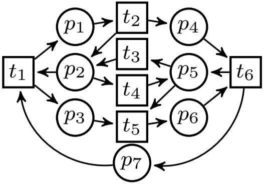
$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}^T \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T$$



$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}^T \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T$$

Example: T-Invariants

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Minimal Invariants

- The *support* of a P-/T-invariant $\mathbf{y}$ is the set $\{i \mid \mathbf{y}[i] > 0\}$. An invariant has *minimal support* iff no invariants support is a strict subset.
  - The number of minimal support P-/T-invariants of a net is finite, but may be exponential.
  - All P-/T-invariants are (positive) linear combinations of minimal support P-/T-invariants.
- A P-/T-invariant $\mathbf{y}$ is *minimal* iff no $\mathbf{y}' < \mathbf{y}$ is invariant.
  - A minimal invariant need not have minimal support.
  - For each minimal support, there is a unique minimal invariant.
- Algorithms exist to generate all minimal support P-/T-invariants of a net.

The Fourier-Motzkin Algorithm for P-Invariants

1. Initialise $B = [W : I_n]$ $(n = |P|)$.
2. For $j = 1, \ldots, |T|$
   1. Append to B all rows resulting from positive linear combinations of pairs of rows in B that eliminate column j.
   2. Remove from B all rows with non-zero jth element.
3. $B = [\mathbf{0} : D]$, where the rows of D are P-invariants.

Example

$$B = \left(\begin{array}{cccccc|ccccccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Example

$$B = \left(\begin{array}{rrrrrr|rrrrrrr}
1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\[1.5em]
0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right)$$

Example

$$B = \left(\begin{array}{cccccc|ccccccc}
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
 & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1
\end{array}\right)$$

Example

$$B = \left(\begin{array}{cccccc|ccccccc}
0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
 & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 2 & 1 & 1 & 1 & 0 & 1 & 0 & 1
\end{array}\right)$$

Example

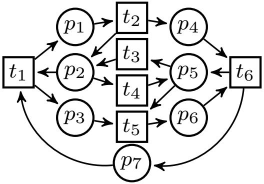
$$B = \left(\begin{array}{cccccc|ccccccc}
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 2 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
 & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 2 & 1
\end{array}\right)$$

Example

$$B = \left(\begin{array}{cccccc|ccccccc} 0&0&0&0&0&0&1&0&0&1&0&0&1 \\ 0&0&0&0&0&0&0&0&1&0&0&1&1 \\ 0&0&0&0&0&0&1&1&0&0&1&1&0 \\ 0&0&0&0&0&0&1&1&1&0&1&2&1 \end{array}\right)$$

P-invariants:

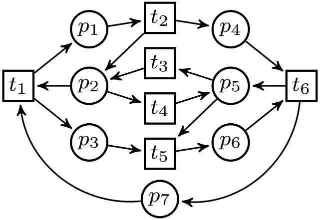
- $\mathbf{z}_1 = (1\,0\,0\,1\,0\,0\,1)$
- $\mathbf{z}_2 = (0\,0\,1\,0\,0\,1\,1)$
- $\mathbf{z}_3 = (1\,1\,0\,0\,1\,1\,0)$
- $\mathbf{z}_4 = (1\,1\,1\,0\,1\,2\,1)$



The State Equation & Structural Properties

- N is structurally bounded iff $\mathbf{y}^T W \leq \mathbf{0}$ has a solution $\mathbf{y} \in \mathbb{N}^{|P|}$ such that $\mathbf{y}[i] \geq 1$ for $i = 1, \ldots, |P|$.
  - $\mathbf{y}$ is a linear combination of *all* place markings that is invariant or decreasing under any transition firing.
- N is repetitive w.r.t. transition t iff $W\mathbf{x} \geq \mathbf{0}$ has a solution $\mathbf{x} \in \mathbb{N}^{|T|}$ such that $\mathbf{x}[t] > 0$.
  - $\mathbf{x}$ is a multiset of transitions, including t at least once, whose combined effect is zero or increasing.
  - Can always find some initial marking $\mathbf{m}_0$ from which $\mathbf{x}$ is realisable.

Example



- $\mathbf{z}_1 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{z}_4 = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 2 & 1 \end{pmatrix}$ are P-invariants of the net.
- $\mathbf{y} = \mathbf{z}_1 + \mathbf{z}_4 = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 2 & 2 \end{pmatrix}$ is also a P-invariant.
- $\mathbf{y}^T W = \mathbf{0}$ and $\mathbf{y} \geq \mathbf{1}$: The net is **structurally bounded.**

Reachability

- Decidability of the (exact) reachability problem for general Petri nets was open for some time.
  - Algorithm proposed by Sacerdote & Tenney in 1977 incorrect (or gaps in correctness proof).
  - Correct algorithm by Mayr in 1981.
  - Simpler correctness proof (for essentially the same algorithm) by Kosaraju in 1982.
- Other algorithms have been presented since.
- All existing algorithms have unbounded complexity.
  - Fun fact: A 2-EXP algorithm was proposed in 1998, but later shown to be incorrect.

Reachability: Preliminaries

- $\mathbf{m}$ is *semi-reachable* from $\mathbf{m}_0$ iff there is a transition sequence $s = t_{i_1}, \ldots, t_{i_n}$ such that $\mathbf{m} = \mathbf{m}_0 + \mathbf{w}(t_{i_1}) + \ldots + \mathbf{w}(t_{i_n})$.
  - s is does not have to be valid (firable) at $\mathbf{m}_0$.
  - $\mathbf{m}$ is semi-reachable from $\mathbf{m}_0$ iff $W\mathbf{n} = (\mathbf{m} - \mathbf{m}_0)$ has a solution $\mathbf{n} \in \mathbb{N}^{|T|}$.
- If $\mathbf{m}$ is semi-reachable from $\mathbf{m}_0$, then $\mathbf{m} + \mathbf{a}$ is reachable from $\mathbf{m}_0 + \mathbf{a}$ for some sufficiently large $\mathbf{a} \geq 0$.

- A *controlled net* is a pair of a marked net $(N = \langle P, T, F\rangle, \mathbf{m}_0)$ and an NFA (A, q_0) over alphabet T.
  - A defines a (regular) subset of (not necessarily firable) transition sequences.
  - Define reachability/coverability/boundedness for $(N, \mathbf{m}_0)$ w.r.t. A in the obvious way.
  - The coverability tree construction is easily modified to consider only sequences accepted by A.
- The *reverse* of N, N_{Rev} (w.r.t. A) is obtained by reversing the flow relation (and arcs in A).
  - $W(N_{\mathsf{Rev}}) = -W(N)$.

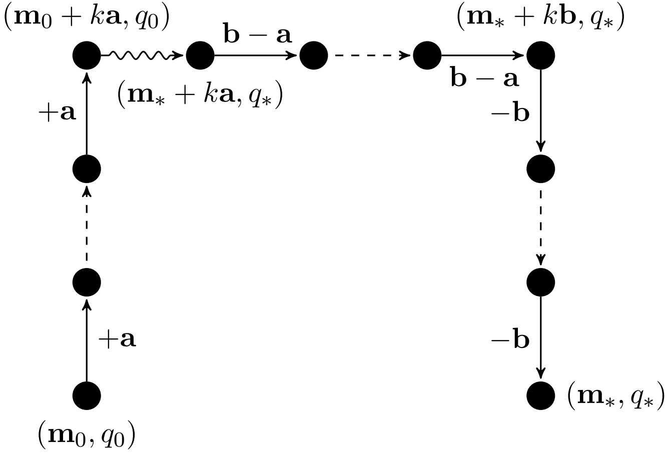
Reachability: A Sufficient Condition

- In $(N, \mathbf{m}_0)$ w.r.t. (A, q_0), if

(a) $(\mathbf{m}_*, q_*)$ is semi-reachable from $(\mathbf{m}_0, q_0)$,
(b) $(\mathbf{m}_0 + \mathbf{a}, q_0)$ is reachable from $(\mathbf{m}_0, q_0)$, for $\mathbf{a} \geq 1$,
(c) $(\mathbf{m}_* + \mathbf{b}, q_*)$ is reachable from $(\mathbf{m}_*, q_*)$ in N_{Rev} w.r.t. A, for $\mathbf{b} \geq 1$,
(d) $(\mathbf{b} - \mathbf{a}, q_*)$ is semi-reachable from $(\mathbf{0}, q_*)$,

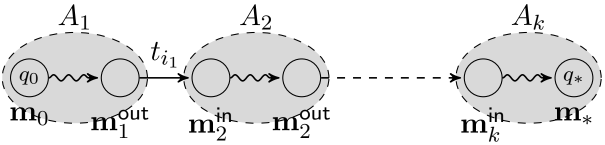
then $(\mathbf{m}_*, q_*)$ is reachable from $(\mathbf{m}_0, q_0)$.

- The conditions above are effectively checkable:
  - (b) & (c) by coverability tree construction,
  - (a) & (d) through the state equation.


$(\mathbf{m}_0 + k\mathbf{a}, q_0)$
$\mathbf{b} - \mathbf{a}$
$(\mathbf{m}_* + k\mathbf{b}, q_*)$
$\mathbf{b} - \mathbf{a}$
$(\mathbf{m}_* + k\mathbf{a}, q_*)$
$+\mathbf{a}$
$-\mathbf{b}$
$+\mathbf{a}$
$-\mathbf{b}$
$(\mathbf{m}_*, q_*)$
$(\mathbf{m}_0, q_0)$

Reachability: The Mayr/Kosaraju Algorithm

Consider a controlled net (N, A) of the form,



with constraints $\mathbf{m}_i^{\text{in/out}}[j] = x_{i,j}^{\text{i/o}}$ or $\mathbf{m}_i^{\text{in/out}}[j] \geq y_{i,j}^{\text{i/o}} \geq 0$.

- If the sufficient reachability condition holds for each $(\mathbf{m}_i^{\text{in}}, q_i^{\text{in}})$ and $(\mathbf{m}_i^{\text{out}}, q_i^{\text{out}})$ w.r.t A_i, then $(\mathbf{m}_*, q_*)$ is reachable from $(\mathbf{m}_0, q_0)$.
- Let $\Delta(A_i) = \{\mathbf{m} \,|\, \mathbf{m} = W\mathbf{n}(s), s \in L(A_i)\}$.
- Let $\Gamma = \{\mathbf{m}_i^{\text{in}}, \mathbf{m}_i^{\text{out}}, \mathbf{n}_i \,|\, \mathbf{m_{i+1}}^{\text{in}} - \mathbf{m_i}^{\text{out}} = \mathbf{w}(t_{i_i}), \mathbf{m_i}^{\text{out}} - \mathbf{m_i}^{\text{in}} \in \Delta(A_i), \text{ and constraints hold}\}$.
- If $(\mathbf{m}_0, q_0)\,[s\rangle\,(\mathbf{m}_*, q_*)$, s defines an element in Γ.

- Γ is a *semi-linear set*: consistency (non-emptiness) is decidable via Pressburger arithmetic.
- If Γ is consistent, but the sufficient condition does not hold in some A_i, then A_i can be replaced by a new "chain" of controllers, $A_i^1, \ldots, A_i^{l_i}$, each of which is "simpler":
  - more equality constraints ($\mathbf{m}_{i^l}^{\text{in/out}} = x_{i^l,j}^{\text{i/o}}$), or
  - same equality constraints and smaller automaton.
- There can be several possible replacements (non-deterministic choice).
- If $(\mathbf{m}_*, q_*)$ is not reachable from $(\mathbf{m}_0, q_0)$, every choice (branch) eventually leads to an inconsistent system.

Special Classes of Nets

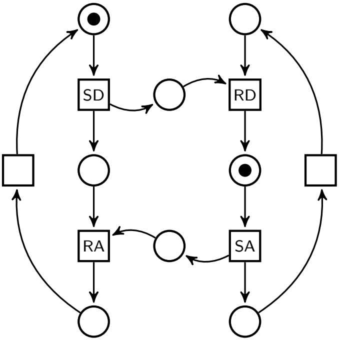
Special Classes of Nets

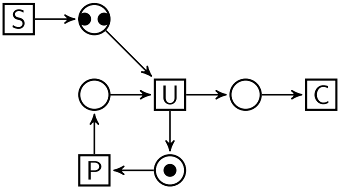
- **State Machines:**
  - every transition has one incoming and one outgoing arc i.e. $|{}^\bullet t| = |t^\bullet| = 1$ for each $t \in T$.
- **Marked Graphs:**
  - every place has one incoming arc, and one outgoing arc i.e. $|{}^\bullet p| = |p^\bullet| = 1$ for each $p \in P$.
- **Free-choice Nets:**
  - every arc is either the only arc going from the place, or only arc going to the transition i.e. $|p^\bullet| \leq 1$ or ${}^\bullet(p^\bullet) = \{p\}$ for each $p \in P$.

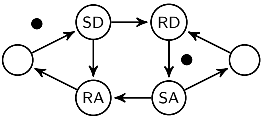
Marked Graphs

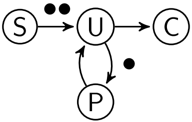
- An ordinary Petri net with $|{}^\bullet p| = |p^\bullet| = 1$ for each place p is a *T-graph*, or *marked graph*.
- Abstracting away places leaves a directed graph:
  - Called the *underlying graph* (usually denoted G).
  - A marking of the net is a marking of the edges of G.
- Marked graphs model "decision-free" concurrent systems.
- Several properties of marked graphs are decidable in polynomial time:
  - Structural liveness and boundedness.
  - Liveness and boundedness for a given initial marking.
- Simple condition for realisability (and thus reachability).

Example: Marked and Underlying Graphs









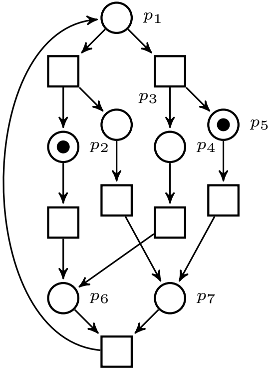
Some Properties of Marked Graphs

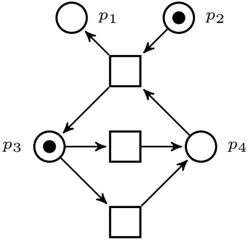
- **Theorem:** The total number of tokens on every directed circuit in the underlying graph is invariant.
- **Theorem:** The maximum number of tokens an edge $a \to b$ in $(G, \mathbf{m}_0)$ can ever have is equal to the minimum number of tokens $\mathbf{m}_0$ places on any directed circuit that contains this edge.
- **Theorem:** A marked graph $(G, \mathbf{m}_0)$ is live iff $\mathbf{m}_0$ places at least one token on every directed circuit of G.
- **Theorem:** A live marked graph $(G, \mathbf{m}_0)$ is k-bounded iff every place (edge in G) belongs to a directed circuit and $\mathbf{m}_0$ places at most k tokens on every directed circuit of G.
- **Theorem:** A marked graph net has a live and bounded marking iff G is strongly connected.

Free Choice Nets

- An ordinary Petri net such that $|p^\bullet| \leq 1$ or ${}^\bullet(p^\bullet) = \{p\}$ for each place p, is a *free choice* net.
  - Equivalently: If $p^\bullet \cap p'^\bullet \neq \emptyset$ then $|p^\bullet| = |p'^\bullet| = 1$, for all $p, p' \in P$.
- Extended free choice net: If $p^\bullet \cap p'^\bullet \neq \emptyset$ then $p^\bullet = p'^\bullet$, for all $p, p' \in P$.
  - An extended free choice net can be transformed to a basic free choice net, adding at most a linear number of places and transitions.
- Note: Marked graphs and state machines are also free choice nets.
- A fundamental property of free choice nets: if ${}^\bullet t \cap {}^\bullet t' \neq \emptyset$ then whenever t is enabled, so is t'.

Example: Free Choice Nets





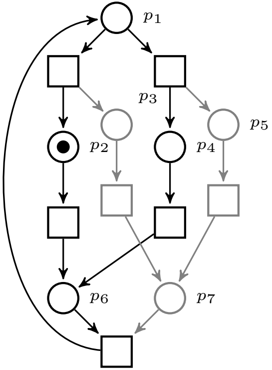
Decomposition of Free Choice Nets

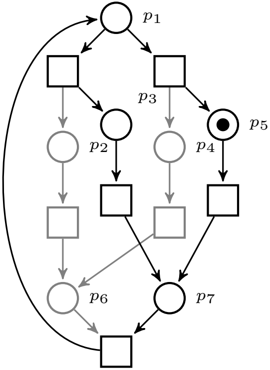
- A subnet of $N = (P, T, W)$ is a net $N' = (P', T', W')$ with $P' \subseteq P$, $T' \subseteq T$ and $W' = W_{|(P' \cup T')}$.
- The P-subnet induced by $S \subseteq P$ is $(S, {}^\bullet S \cup S^\bullet, W')$
  - That is, the subnet consisting of S and all transition incident on a place in S.
- The T-subnet induced by $U \subseteq T$ is $({}^\bullet U \cup U^\bullet, U, W')$
  - That is, the subnet consisting of U and all places incident on a transition in U.
- A *P-component* is a strongly connected P-subnet such that $|{}^\bullet t|, |t^\bullet| \leq 1$, for all t.
  - A P-component is a state machine.
- A *T-component* is a strongly connected T-subnet such that $|{}^\bullet p|, |p^\bullet| \leq 1$, for all p
  - A T-component is a marked graph.

Decomposition of Free Choice Nets

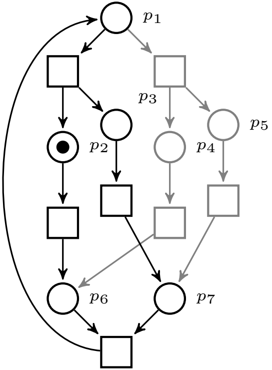
- **Theorem:** A live free choice net $(N, \mathbf{m}_0)$ is 1-bounded (safe) iff it is covered by P-components, each of which has a single token at $\mathbf{m}_0$.
- **Theorem:** A live and safe free choice net $(N, \mathbf{m}_0)$ is covered by T-components, and for each T-component, N', there is a reachable marking $\mathbf{m}$ such that $(N', \mathbf{m}_{|N'})$ is live and safe.

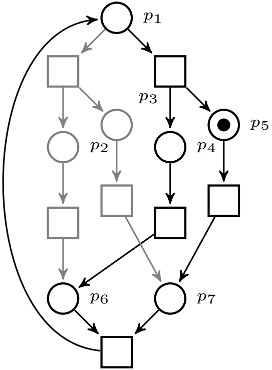
Example: Decomposition into State Machines





Example: Decomposition into Marked Graphs

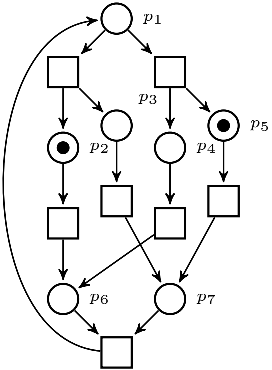


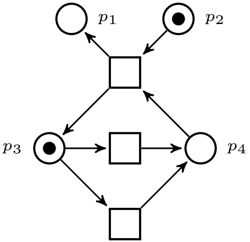


Siphons and Traps

- A *siphon* is a subset S of places such that ${}^{\bullet}S \subseteq S^{\bullet}$.
  - Every transition that outputs a token to a place in S also consumes a token from a place in S.
  - If $\mathbf{m}$ places no token in S, no marking reachable from $\mathbf{m}$ does either.
- A *trap* is a subset S of places such that $S^{\bullet} \subseteq {}^{\bullet}S$.
  - Every transition that consumes a token from a place in S also outputs a token to a place in S.
  - If $\mathbf{m}$ places at least one token in S, so does every marking reachable from $\mathbf{m}$.
- **Theorem:** A free choice net $(N, \mathbf{m}_0)$ is live iff every siphon contains a marked trap.

Example: Siphons and Traps





- The only siphon is P.
- P is also a trap.

- $\{p_2\}$ and $\{p_3, p_4\}$ are siphons.
- $\{p_1\}$ and $\{p_3, p_4\}$ are traps.

Some Complexity Results for Free Choice Nets

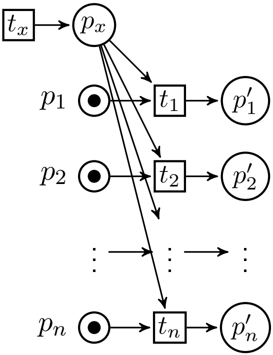
- Liveness for marked free choice nets is decidable in polynomial time.
- Boundedness of *live* free choice nets is decidable in polynomial time.
- A number of properties of *live and bounded* free choice nets are decidable in polynomial time, e.g.,
  - Transition executability and repeated executability.
  - The "home state" property (markings that can always be re-reached).
- Reachability in free choice nets is NP-hard.

Characterisation by Derivation Rules

- Initial net: 
- Rule #1: Add a new place p' with $\mathbf{r}(p') = \sum_{p \in P} \lambda_p \mathbf{r}(p)$ and $|p'^{\bullet}| = 1$.
- Rule #2: Replace place p with a connected P-graph N', and connect each input and output of p to at least one place in N'.
  - Must have $|{}^{\bullet}p| > 1$ and $|p^{\bullet}| > 1$, except for initial net.
  - Every place $p' \in N'$ must appear on a path in the resulting net that enters and leaves N'.
- **Theorem:** The class of nets obtained by applying the above rules to the initial net is exactly the class of structurally live and structurally bounded free choice nets.

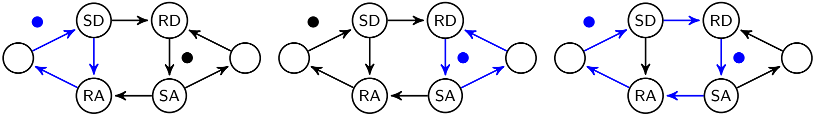
Reachability: Acyclic Nets

- Recall: $\mathbf{m}_0\,[s\rangle\,\mathbf{m}$ implies $\exists \mathbf{n} \in \mathbb{N}^{|T|} : W\mathbf{n} = (\mathbf{m} - \mathbf{m}_0)$.
- A solution $\mathbf{n}$ is *realisable* iff $\mathbf{n} = \mathbf{n}(s)$ for some valid firing sequence s.
- **Theorem:** For an acyclic net, every solution to $W\mathbf{n} = (\mathbf{m} - \mathbf{m}_0)$ is realisable.
- Reachability in acyclic nets is NP-hard.



Reachability: Marked Graphs

- **Theorem:** In a live marked graph, $\mathbf{m}$ is reachable from $\mathbf{m}_0$ iff $\mathbf{m}_0$ and $\mathbf{m}$ place the same total number of tokens on every fundamental circuit of the underlying graph.
  - A fundamental circuit is obtained by adding one edge to a spanning tree.
  - The directed fundamental circuits of a marked graph are a full set of linearly independent P-invariants.



Conclusions

Summary & Conclusions

- Petri nets: Intuitive, graphical modelling formalism, closely related to planning.
- Petri net theory offers a different set of tools:
  - Algebraic methods (based on the state equation).
  - Characterisation and study of classes of nets with special structure.
- Planning also has tools potentially applicable to Petri nets.

The Many Things We Haven't Talked About

- Extensions of basic Place-Transition nets:
  - Read arcs, reset arcs and inhibitor arcs.
  - Colored Petri nets, timed nets, stochastic nets, etc.
- Other properties of Petri nets (and related decision problems):
  - Model checking (tense logics, process calculi).
  - Language (trace) properties.
- Heaps more results concerning different Petri net subclasses.